Solution

1. **Method 1:** Deriving the governing EOM using EL equations
   The potential energy:

$$V = \frac{1}{2}kx_1^2 + \frac{3}{2}kx_4^2 + k(x_4 - x_3)^2 - mgx_1 - mgx_2 - \frac{7}{4}mgx_3 - 6mgx_4 \tag{1}$$

   The kinetic energy:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{7}{8}m\dot{x}_3^2 + 3m\dot{x}_4^2 + m\dot{\theta}_1^2 + m\dot{\theta}_2^2 \tag{2}$$

   Rayleigh function:

$$D = \frac{1}{2}c\dot{x}_1^2 + \frac{1}{8}c\dot{x}_3^2 + 5c\dot{x}_4^2 \tag{3}$$

   Substituting the kinematic conditions:

$$\begin{aligned} V &= 18k\theta_1^2 + \frac{5}{2}kx_4^2 - 8kx_4\theta_1 - 11mg\theta_1 - 6mgx_4 \\ T &= 20m\dot{\theta}_1^2 + 3m\dot{x}_4^2 \\ D &= 4c\dot{\theta}_1^2 + 5c\dot{x}_4^2 \end{aligned} \tag{4}$$

Employing EL equations

$$\begin{aligned} &\mathcal{L} = T - V \\ &\frac{\partial \mathcal{L}}{\partial \theta_1} = -36k\theta_1 + 8kx_4 + 11mg, \quad \frac{\partial \mathcal{L}}{\partial x_4} = -5kx_4 + 8k\theta_1 + 6mg \\ &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = 40m\ddot{\theta}_1, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_4}\right) = 6m_4\ddot{x}_4 \\ &\frac{\partial D}{\partial \dot{\theta}_1} = 8c\dot{\theta}_1, \quad \frac{\partial D}{\partial \dot{x}_1} = 10c\dot{x}_4 \end{aligned}$$

The governing equations are

$$\begin{aligned} 40m\ddot{\theta}_1 + 8c\dot{\theta}_1 + 36k\theta_1 - 8kx_4 - 11mg &= T \\ 6m\ddot{x}_4 + 10c\dot{x}_4 + 5kx_4 - 8k\theta_1 - 6mg &= F \end{aligned} \tag{5}$$

2. To find the equilibrium point we set $\ddot{\theta}_1 = \dot{\theta}_1 = \ddot{x}_4 = \dot{x}_4 = F = T = 0$ to the equation of motion

$$\begin{aligned} 36k\theta_1 - 8kx_4 - 11mg &= 0 \\ 5kx_4 - 8k\theta_1 - 6mg &= 0 \end{aligned} \tag{6}$$

Solving the two equations leads to the following:

$$\{\theta_1, x_4\}_{eq} = \left\{\frac{103mg}{116k}, \frac{76mg}{29k}\right\} = \left\{0.8879\frac{mg}{k}[\text{rad}], 2.6207\frac{mg}{k}[\text{m}]\right\} \tag{7}$$

3. To compute the transfer function we employ a Laplace transform the equations of motion

$2\xi\omega_n = \frac{c}{m}$

$$\begin{cases}\left(40ms^2+8cs+36k\right)\Theta_1-8kX_4=\tilde{T}\Rightarrow & \Theta_1=\dfrac{\tilde{T}+8kX_4}{40ms^2+8cs+36k}\\ \left(6ms^2+10cs+5k\right)X_4-8k\Theta_1=\tilde{F}\end{cases}$$

$$\left[\frac{\left(6ms^2+10cs+5k\right)\left(40ms^2+8cs+36k\right)-64k^2}{40ms^2+8cs+36k}\right]X_4=\tilde{F}+\frac{8k}{40ms^2+8cs+36k}\tilde{T}$$

Therefore, the first transfer function is:

$$G_1(s)=\frac{X_4}{\tilde{T}}=\frac{8k}{\left(6ms^2+10cs+5k\right)\left(40ms^2+8cs+36k\right)-64k^2} \tag{8}$$

And the second transfer function is:

$$\frac{V_4}{\tilde{F}}=\frac{40ms^3+8cs^2+36ks}{\left(6ms^2+10cs+5k\right)\left(40ms^2+8cs+36k\right)-64k^2} \tag{9}$$

4. Rearranging transfer function to the standard from:

$$G(s)=\frac{1/20}{\left(s^2+s+625\right)\left(s^2+368s+52900\right)} \tag{10}$$

Now, we can find the poles of the transfer function, or directly the natural frequencies and damping coefficients:

$$\begin{aligned}\omega_1&=\sqrt{625}=25\,\text{rad/s},\quad \zeta_1=\frac{1}{50}=0.02\\ \omega_2&=\sqrt{52900}=230\,\text{rad/s},\quad \zeta_2=\frac{368}{460}=0.8\end{aligned} \tag{11}$$

For the first frequency, we should expect a resonance peak and for the second not (>0.707), therefore, the correct plot is (a).

5) For the input torque at 7000 rad/s the expected phase for x_3 should be close to -2π, because the system has two sets of conjugate poles at 25 rad/s and 230 rad/s, and each adds a phase shift of $-\pi$. But the question is about $\dot{x}_3$, and the time derivative adds $\pi/2$ to the phase, so the total phase shift is expected to be $-3\pi/2$.

-270

Question 2

1) $(x,y)_M = (z\cos\beta, -z\sin\beta)$ (6pts)

$(x,y)_m = (z\cos\beta + l\sin\theta, -z\sin\beta - l\cos\theta)$

$V_M^2 = \dot{z}^2$

$V_m^2 = \dot{z}^2 + l^2\dot{\theta}^2 + 2l\dot{z}\dot{\theta}(\cos\beta\cos\theta - \sin\beta\sin\theta)$

Lagrangian:

$$L = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{z}^2 + l^2\dot{\theta}^2 + 2l\dot{z}\dot{\theta}\cos(\theta+\beta)\right) + Mgz\sin\beta + mg(z\sin\beta + l\cos\theta)$$

Equations of motion

$$(M+m)\ddot{z} + ml\left(\ddot{\theta}\cos(\theta+\beta) - \dot{\theta}^2\sin(\theta+\beta)\right) = (M+m)g\sin\beta$$

$$l\ddot{\theta} + \ddot{z}\cos(\theta+\beta) = -g\sin\theta$$

2) Equilibrium point $\ddot{\theta} = \dot{\theta} = 0$ (4pts)

Eq(1) ; $\ddot{z} = g\sin\beta$

Eq(2) ; $g\sin\beta\cos(\theta+\beta) = -g\sin\theta$

$\tan\theta = -\tan\beta$ $\quad \theta = -\beta$ or $\theta = \pi - \beta$

stable $\quad$ unstable

Problem 3

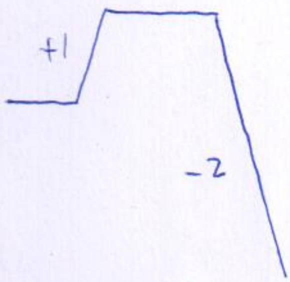
(4 pts)

1) $$G(s) = \frac{A(\tau_1 s+1)}{(\tau_2 s+1)(\tau_3 s+1)(\tau_4 s+1)} \qquad \tau_1 > \tau_2 > \tau_3 > \tau_4$$

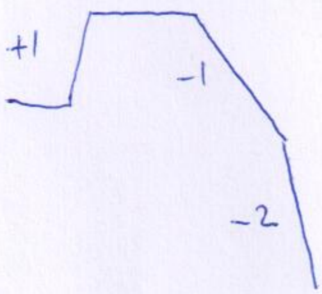
(+1 bonus)

$n = 3$ (1 pt)

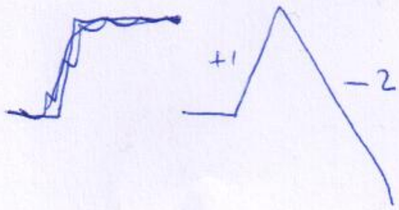
2) (2pts) one repeated



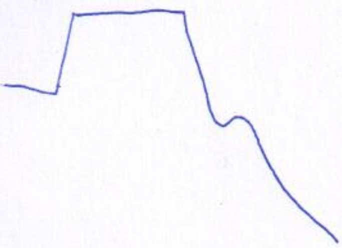
(3pts) all distinct



(2pts) all repeated



(3 pts) one complex conjugate



Problem 4

a)

$$G(s) = \frac{s^4+2s^3+s^2+2}{s^3+2s^2+s} = s + \frac{2}{s(s+1)^2}$$

5 pt

$$= s + \frac{2}{s} + \frac{-2}{(s+1)^2} + \frac{-2}{s+1}$$

$$= \delta'(t) + (2 - 2te^{-t} - 2e^{-t})\,\varepsilon(t)$$

2 pt

b) 1. $G_1(s) = \dfrac{5}{10s+1}$ (4 pt) $\tau_1 = 10$

$t=\tau \quad y(\tau) = 0{,}63\,K$

$t=3\tau \quad y(3\tau) = 0{,}95\,k$

where $K=5$

$G_2(s) = \dfrac{20}{\cancel{s+10}\ \ 0.1s+1}$ (4 pt) $\frac{1}{\tau_2}$ = corner frequency where $|G(j\omega)| = \frac{K}{\sqrt{2}}$

$K=20$

2. $P_1 = -0.1$ (1 pt) $P_2 = -10$ → First system (1 pt)

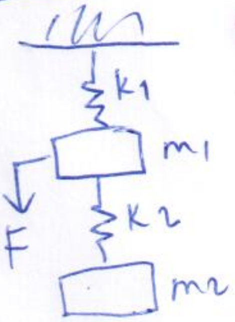
3. $u = e^{-10t}$ $\quad U(s) = \dfrac{1}{s+10}$ (2 pts) $\quad Y(s) = \dfrac{200}{(s+10)^2} = \cancel{\frac{\ }{(s+10)^2}\ \frac{\ }{s+10}}$

↓

$y(t) = 200\,t\,e^{-10t}$ (3 pts)

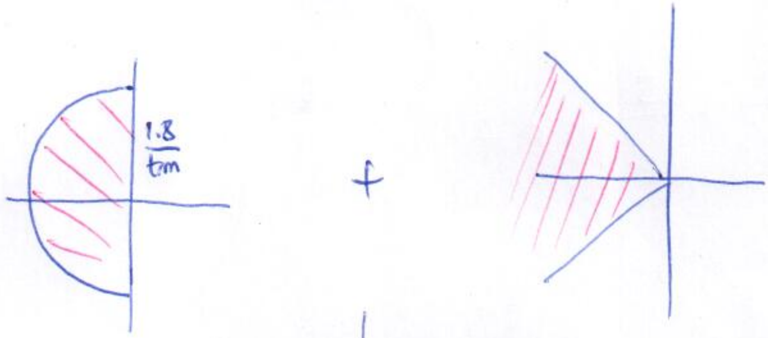
Analogue

(3 points)

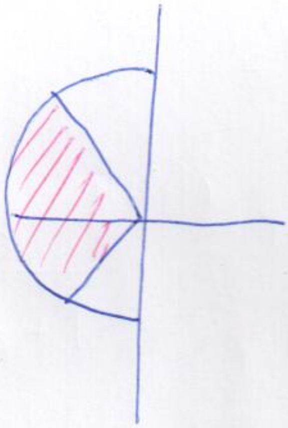


Problem 5

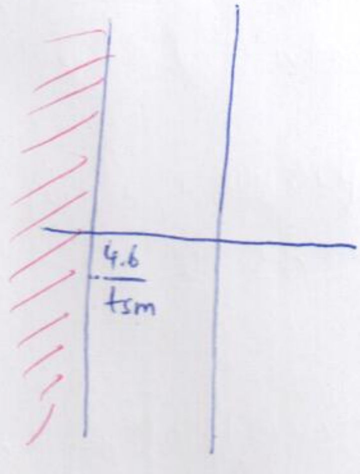
a) (i) $\omega_0 < \dfrac{1.8}{t_{rm}}$ and $\zeta > 0.6\,(1 - M_{max})$



(3pts)



(ii) $\zeta \omega_n > \dfrac{4.6}{t_{sm}}$



(3pts)

b)

(i) same (3pts)

(ii) different (3pts)

how? → bonus (2pts)

c) no resonance (3pts)

d) $F(s) = \dfrac{0.1s + 1}{s}$ (10pts)